

## ON LOCAL CATEGORIES OF FINITE GROUPS II

FEI XU

ABSTRACT. Local categories of a finite group  $G$  are considered as generalized subquotient groups, and their representations contribute to group representations. To show this, we investigate the (stable) Grothendieck groups of the relevant local category algebras, as well as several natural maps among them. In this way, we may realize homology representations of  $G$  via a generalized induction process. We also provide a new way to reformulate the Alvis-Curtis duality when  $G$  is Chevalley.

## 1. INTRODUCTION

Let  $G$  be a finite group and  $\mathcal{P}$  be a finite  $G$ -poset. We are interested in the finite (abstract) transporter category  $\mathcal{P} \rtimes G$ , and its various quotient categories  $\mathcal{C}$ , which will be called the *local categories* of  $G$  [8]. When  $\mathcal{P} = \bullet$  is a trivial poset with trivial action,  $\bullet \rtimes G \cong G$  is a special case. Other typical examples include the  $p$ -transporter category  $\mathrm{Tr}_p(G) = \mathcal{S}_p \rtimes G$  and the  $p$ -orbit category  $\mathcal{O}_p(G)$ . We shall regard transporter categories as generalized subgroups, and the other local categories as generalized subquotient groups, of  $G$ .

To carry on our tasks, we shall deal with the category algebra  $R\mathcal{C}$ , where  $R$  is a commutative ring with identity, and the category  $\mathrm{mod}\text{-}R\mathcal{C}$  of finitely generated *right* modules (equivalently, contravariant functors from  $\mathcal{C}$  to the category of finitely generated  $R$ -modules). The module category is known to be symmetric monoidal, and the tensor structure easily lifts to  $D^b(\mathrm{mod}\text{-}R\mathcal{C})$ , the bounded derived category. In our earlier work on category algebras, we used to work with left modules. Here we turn to right modules, because we shall consider the so-called *coefficient systems*, which are by definition the right modules of  $R\mathcal{O}_p(G)$  [5].

Let  $R$  be a field. It is known, for a finitely generated  $R$ -algebra  $A$ , that  $G_0(D^b(\mathrm{mod}\text{-}A))$  (the Grothendieck group of a triangulated category) is isomorphic to  $G_0(\mathrm{mod}\text{-}A)$  (the Grothendieck group of an Abelian category) [12]. For our convenience, we shall denote both by

---

*Key words and phrases.*  $G$ -poset, local category, category algebra, Kan extension, (stable) Grothendieck ring, homology representation.

The author (徐斐) is supported by the NSFC grant No. 11671245.

$G_0(A)$ , called the *Grothendieck group* of  $A$ . We are interested in the case of  $A = RC$  for a local category  $\mathcal{C}$  of  $G$ . Under the circumstance,  $G_0(RC)$  has a natural ring structure, and will be called the *Grothendieck ring* of  $RC$ . At this point, let us focus on the transporter category  $\mathcal{P} \rtimes G$ . Since  $R\mathcal{P} \rtimes G$  is Gorenstein [10], we may continue to consider the stable category  $\underline{\mathbf{CM}}(R\mathcal{P} \rtimes G)$ , of maximal Cohen-Macaulay modules, which comes from a localization sequence of tensor triangulated categories

$$D^b(\text{proj-}R\mathcal{P} \rtimes G) \rightarrow D^b(\text{mod-}R\mathcal{P} \rtimes G) \rightarrow \underline{\mathbf{CM}}(R\mathcal{P} \rtimes G)$$

and results in a short exact sequence of Abelian groups

$$0 \rightarrow G_0(D^b(\text{proj-}R\mathcal{P} \rtimes G)) \xrightarrow{c} G_0(R\mathcal{P} \rtimes G) \rightarrow G_0^{st}(R\mathcal{P} \rtimes G) \rightarrow 0.$$

Here  $G_0^{st}(R\mathcal{P} \rtimes G) = G_0(\underline{\mathbf{CM}}(R\mathcal{P} \rtimes G))$  is the *stable Grothendieck ring*, and  $c$  is the Cartan map. The above maps are actually ring homomorphisms. Moreover since any module tensoring with a module of finite projective dimension is still of finite projective dimension,  $G_0(D^b(\text{proj-}R\mathcal{P} \rtimes G))$  becomes a (two-sided) ideal of  $G_0(R\mathcal{P} \rtimes G)$ .

The stable Grothendieck group is interesting only when the characteristic of  $R$  divides the order of  $G$ , for otherwise  $G_0(D^b(\text{proj-}R\mathcal{P} \rtimes G)) = G_0(R\mathcal{P} \rtimes G)$ . As we mentioned earlier, if  $\mathcal{P} = \bullet$ , then  $R \bullet \rtimes G \cong RG$ . When  $p$  is of characteristic  $R$  that divides the order of  $G$ ,  $\underline{\mathbf{CM}}(RG)$  is commonly written as  $\underline{\mathbf{mod}}\text{-}RG$  or  $\text{stmod}RG$ .

The main theme of our work is to study  $G_0(RC)$  (or  $G_0^{st}(RC)$ , if applicable), for a local category  $\mathcal{C}$ , and establish their connections with  $G_0(RG)$  or  $G_0^{st}(RG)$  ( $R$  ranging through the three rings in a suitable  $p$ -modular system  $(K, \mathcal{O}, k)$  where  $p \mid |G|$ ). We shall demonstrate that certain interesting maps are given by a generalized restriction and a generalized induction. Consequently, the *homology representation* of  $G$  on  $\mathcal{P}$ ,  $\sum_i (-1)^i [H_i(\mathcal{P}, R)] \in G_0(RG)$ , is “induced” from the trivial representation  $[\underline{R}] \in G_0(R\mathcal{P} \rtimes G)$ , which in turn is the restriction of  $[R] \in G_0(kG)$  (one may consider this as an “algebraization” of homology representations).

As another example (Theorem 5.1), when  $\mathcal{C} = \mathcal{O}_{\mathcal{B}_p}(G)$  is the orbit category of a Chevalley group  $G$ , in defining characteristic  $p$ , built on the unipotent radicals of parabolic subgroups, then we have maps

$$G_0(\mathbb{C}G) \xrightarrow{t_G} G_0(\mathbb{C}\mathcal{O}_{\mathcal{B}_p}(G)) \xrightarrow{i_G} G_0(\mathbb{C}G)$$

such that  $i_G t_G - \text{Id}$  is the Alvis-Curtis duality  $D_G$ . Here  $t_G$  is induced by a fixed point coefficient system, and  $i_G$  by the homology representation on  $\mathcal{B}_p$  (a  $G$ -subposet of  $\mathcal{S}_p$ ).

## 2. LOCAL CATEGORIES

The present paper may be considered as a continuation of [8]. Thus let us recall some fundamental constructions from there. Let  $G$  be a finite group and  $\mathcal{P}$  be a finite  $G$ -poset (e.g.  $\mathcal{S}_p$ , the poset of non-trivial  $p$ -subgroups of  $G$  and  $\mathcal{S}_p^1 = \mathcal{S}_p \cup \{1\}$ ). Let  $x \in \text{Ob } \mathcal{P}$  and  $\alpha \in \text{Mor } \mathcal{P}$ . We shall denote by  ${}^g x \in \text{Ob } \mathcal{P}$  and  ${}^g \alpha \in \text{Mor } \mathcal{P}$  their images under the action by an element  $g \in G$ . We use 1 for the identity of  $G$ . Its action on  $\mathcal{P}$  is always the trivial.

We want to focus on the transporter category  $\mathcal{P} \rtimes G$  and its quotients  $\mathcal{C}$  which are called the *local categories* of  $G$ . In [8], the transporter category was written as  $G \ltimes \mathcal{P}$ . However gradually we realized that  $\mathcal{P} \rtimes G$  is a better notation, emphasizing its nature of being a semi-direct product of categories.

The transporter category  $\mathcal{P} \rtimes G$  has the same objects as  $\mathcal{P}$ . Meanwhile its morphisms are of the form  $\alpha g$ , for  $\alpha \in \text{Mor } \mathcal{P}$  and  $g \in G$ . If  $\beta h$  is another morphism that can be composed with  $\alpha g$ , then  $(\alpha g)(\beta h) := (\alpha {}^g \beta)(gh)$ . In fact, these two morphisms are composable in  $\mathcal{P} \rtimes G$  if and only if  $\alpha$  and  ${}^g \beta$  are composable in  $\mathcal{P}$ . For instance,  $\alpha g \in \text{Hom}_{\mathcal{P} \rtimes G}(x, y)$  itself is a composite  $(\alpha 1)(1_{g_x} g)$ , of  $1_{g_x} g : x \rightarrow {}^g x$  and  $\alpha 1 : {}^g x \rightarrow y$ . Thus  $\alpha g$  may be understood as a “conjugation” followed by an “inclusion”. (Be aware that in general  $(1_{g_x} g)(\alpha 1) \neq \alpha g$ . The former equals  ${}^g \alpha g$ .)

A quotient  $\mathcal{C}$  of  $\mathcal{P} \rtimes G$  comes from a category extension [6]

$$\mathcal{K} \rightarrow \mathcal{P} \rtimes G \rightarrow \mathcal{C}.$$

Here  $\mathcal{K}$  is by definition a functor from  $\mathcal{P} \rtimes G$  to the category of groups. Using axioms, it is easy to identify  $\mathcal{K}$  with a groupoid and a subcategory of  $\mathcal{P} \rtimes G$ . In [8], we pointed out that our interests come from the following diagram

$$\begin{array}{ccc} \mathcal{K} & & \\ & \searrow & \\ & \mathcal{P} \rtimes G & \\ & \swarrow & \searrow \\ G & & \mathcal{C} \end{array}$$

which visualizes connections between the representations of  $RG$  and of  $R\mathcal{C}$  ( $R$  is a commutative ring with identity). In this paper, we use the language of (stable) Grothendieck rings to further demonstrate these connections. In the above diagram,  $\mathcal{P} \rtimes G$  is considered as a

generalized subgroup, while  $\mathcal{C}$  is considered as a generalized subquotient group, of  $G$ .

Given a finite category  $\mathcal{C}$ , we shall investigate its representations through the category algebra  $R\mathcal{C}$  [6, 9]. It has a trivial module  $\underline{k}$ , which is the same as the constant functor from  $\mathcal{C}$  which send every object to  $R$  and every morphism to  $\text{Id}_R$ . Recall that there exists an object-wise tensor product, written as  $\hat{\otimes}$ , between functors. It makes  $\text{mod-}k\mathcal{C}$  (and hence  $D^b(\text{mod-}R\mathcal{C})$ ) into a symmetric monoidal category with tensor identity the trivial module  $\underline{k}$ . When  $\mathcal{C} = \mathcal{P} \rtimes G$ , there is a canonical isomorphism  $R(\mathcal{P} \rtimes G) \cong (R\mathcal{P}) \rtimes G$  (the latter being the skew group algebra built on  $R\mathcal{P}$ ). Hence we shall write the transporter category algebra as  $R\mathcal{P} \rtimes G$ . When  $R$  is a field, this is a Gorenstein algebra, because both  $R\mathcal{P}$  and  $RG$  are. It is the reason why we may consider the maximal Cohen-Macaulay modules [10]. For a Gorenstein algebra, a module is of finite projective dimension if and only if it is of finite injective dimension. If  $\mathfrak{M} \in \text{mod-}R\mathcal{P} \rtimes G$ , then  $\mathfrak{M}$  is of finite projective dimension if and only if  $\mathfrak{M}(x), \forall x$ , is of finite projective dimension as a  $RG_x$ -module, where  $G_x \subset G$  is the automorphism group of  $x \in \text{Ob } \mathcal{P} \rtimes G$  (or equivalently the stabilizer of  $x$ ) [8].

### 3. COMPARING GROTHENDIECK RINGS

Let  $R$  be an (algebraically closed) field. Suppose  $A$  is a finite-dimensional  $R$ -algebra. We consider  $\text{mod-}A$ , and  $D^b(\text{mod-}A)$ , the bounded derived category of finitely generated right  $A$ -modules. Our notations about triangulated categories and Grothendieck groups follow [12, Section 6.8]. For an object  $C_* \in D^b(\text{mod-}A)$ , it is well-known that in  $G_0(A)$

$$[C_*] = \sum_i (-1)^i [C_i] = \sum_i (-1)^i [H_i(C_*)].$$

Now assume  $A = R\mathcal{P} \rtimes G$ . A right  $R\mathcal{P} \rtimes G$ -module is the same as a  $G$ -equivariant presheaf on  $\mathcal{P}$  (see [5, 7], sometimes dubbed as a weak coefficient system). From [8], there exists a canonical (covariant) functor

$$\pi : \mathcal{P} \rtimes G \rightarrow G.$$

This functor sends every objects of  $\mathcal{P} \rtimes G$  to the unique object of  $G$  (considered as a category), and maps the morphism  $\alpha g$  to  $g$ . It induces a *restriction*  $\text{Res}_\pi : \text{mod-}RG \rightarrow \text{mod-}R\mathcal{P} \rtimes G$  and we shall denote the values by  $\kappa_M = \text{Res}_\pi M$  (a constant presheaf), for each  $M \in \text{mod-}RG$ . When  $M = R$ , we write  $\underline{R} = \kappa_R$ . The restriction is equipped with two

adjoint functors [8]

$$LK_\pi, RK_\pi : \text{mod-}R\mathcal{P} \rtimes G \rightarrow \text{mod-}RG,$$

called the left and right *Kan extensions*. Suppose  $\mathfrak{M} \in \text{mod-}R\mathcal{P} \rtimes G$ . Then

$$LK_\pi \mathfrak{M} \cong \varinjlim_{\mathcal{P}} \mathfrak{M} \quad \text{and} \quad RK_\pi \mathfrak{M} \cong \varprojlim_{\mathcal{P}} \mathfrak{M}.$$

We shall be mostly dealing with the left Kan extension, as for the right Kan extension, discussion is analogous. When  $\mathcal{P} = G/H$  (with left multiplication action by group elements), for some subgroup  $H$ , the restriction can be identified with  $\downarrow_H^G$ , and the left and right Kan extensions are the induction  $\uparrow_H^G$  and coinduction  $\uparrow_H^G$ .

The derived functors of the Kan extensions are the higher limits, which are isomorphic to the category (co)homology  $\varinjlim_{\mathcal{P}}^i \mathfrak{M} = H_i(\mathcal{P}; \mathfrak{M})$  and  $\varprojlim_{\mathcal{P}}^i \mathfrak{M} = H^i(\mathcal{P}; \mathfrak{M})$ . By definition, the groups  $H_*(\mathcal{P}; \mathfrak{M})$  are the simplicial homology groups of  $\mathcal{P}$  with coefficients in  $\mathfrak{M}$ . In other words, they come from a (finite) complex of  $RG$ -modules  $C_*(\mathcal{P}; \mathfrak{M})$  such that

$$C_i(\mathcal{P}; \mathfrak{M}) = \bigoplus_{x_0 \rightarrow \cdots \rightarrow x_i} \mathfrak{M}(x_0).$$

Here  $x_0, \dots, x_i$  are objects of  $\mathcal{P}$ ,  $x_0 \rightarrow \cdots \rightarrow x_i$  is called an  $i$ -chain, and no objects in a chain repeat themselves. The cohomology groups  $H^i(\mathcal{P}; \mathfrak{M})$  are defined analogously. When  $\mathfrak{M} = \underline{R}$ ,  $C_*(\mathcal{P}; \underline{R})$  equals  $C_*(B\mathcal{P}, R)$ , for the (finite)  $G$ -CW-complex  $B\mathcal{P}$  (the classifying space of  $\mathcal{P}$ ). Hence  $H_i(\mathcal{P}; \underline{R})$  is isomorphic to  $H_i(B\mathcal{P}, R)$ . Note that  $C_i(\mathcal{P}; \underline{R})$  consists of permutation modules, because the set of  $i$ -simplices of  $B\mathcal{P}$  is a  $G$ -set, for each  $i$ . A key observation is that, since  $\mathcal{P}$  is finite,  $\varinjlim_{\mathcal{P}}^i \mathfrak{M} = H_i(\mathcal{P}; \mathfrak{M})$  and  $\varprojlim_{\mathcal{P}}^i \mathfrak{M} = H^i(\mathcal{P}; \mathfrak{M})$  always vanish. It enables us to understand the following constructions on the level of derived categories.

By definition, the restriction is an exact functor. While the left Kan extension is right exact and preserves projectives. They induce two triangulated functors [10]

$$\mathbb{R}\text{es}_\pi : D^b(\text{mod-}RG) \rightarrow D^b(\text{mod-}R\mathcal{P} \rtimes G)$$

and

$$\mathbb{L}K_\pi : D^b(\text{mod-}R\mathcal{P} \rtimes G) \rightarrow D^b(\text{mod-}RG).$$

Since  $\mathbb{R}\text{es}_\pi$  maps  $D^b(\text{proj-}RG)$  into  $D^b(\text{proj-}R\mathcal{P} \rtimes G)$  (complexes of modules with finite projective dimensions), and  $\mathbb{L}K_\pi$  does the converse, they produce functors between the stable categories

$$\mathbb{R}\text{es}'_\pi : \underline{\text{mod-}}RG \rightarrow \underline{\text{CM}}(R\mathcal{P} \rtimes G)$$

and

$$\mathbb{L}K'_\pi : \underline{\mathbf{CM}}(R\mathcal{P} \rtimes G) \rightarrow \underline{\mathbf{mod}}\text{-}RG.$$

Subsequently they give rise to maps between the (stable) Grothendieck groups

$$r_\pi : G_0(RG) \rightarrow G_0(R\mathcal{P} \rtimes G),$$

$$r'_\pi : G_0^{st}(RG) \rightarrow G_0^{st}(R\mathcal{P} \rtimes G),$$

$$lk_\pi : G_0(R\mathcal{P} \rtimes G) \rightarrow G_0(RG),$$

$$lk'_\pi : G_0^{st}(R\mathcal{P} \rtimes G) \rightarrow G_0^{st}(RG).$$

Taking the tensor structures into account, it is obvious that both  $r_\pi$  and  $r'_\pi$  are ring homomorphisms. We may explicitly write out the map  $lk_\pi$  (and  $lk'_\pi$ ) by

$$[\mathfrak{M}] \mapsto \sum_i (-1)^i [\mathrm{H}_i(\mathcal{P}; \mathfrak{M})],$$

for each  $\mathfrak{M} \in \mathbf{mod}\text{-}k\mathcal{P} \rtimes G$ .

There are also analogously defined maps induced by the right Kan extension

$$rk_\pi : G_0(R\mathcal{P} \rtimes G) \rightarrow G_0(RG),$$

and

$$rk'_\pi : G_0^{st}(R\mathcal{P} \rtimes G) \rightarrow G_0^{st}(RG).$$

#### 4. HOMOLOGY REPRESENTATIONS

In this section, we describe an induction method to construct homology representations, using the left Kan extension. It should serve as a motivation for our further investigations. Be aware that in other places, homology representations and the (generalized) Steinberg representation are constructed in  $a(RG)$ , the representation ring (see for instance [1, 3]). Here we study them in  $G_0(RG)$ , through the canonical surjection  $a(RG) \rightarrow G_0(RG)$ .

**Definition 4.1.** Let  $R$  be a commutative ring with identity. We call  $\mathrm{H}_\mathcal{P}(G; \mathfrak{M}) = \sum_i (-1)^i [\mathrm{H}_i(\mathcal{P}; \mathfrak{M})] \in G_0(RG)$  the *homology representation* of  $G$  on  $\mathcal{P}$  with coefficients in  $\mathfrak{M} \in \mathbf{mod}\text{-}R\mathcal{P} \rtimes G$ , and  $\mathrm{St}_\mathcal{P}(G) = \mathrm{H}_\mathcal{P}(G; \underline{R}) - [R]$  the *generalized Steinberg representation* of  $G$  on  $\mathcal{P}$ .

We shall abbreviate  $H_{\mathcal{P}}(G; \underline{R})$  as  $H_{\mathcal{P}}(G)$ .

Note that in [3],  $\bigoplus_i H_i(\mathcal{P}; \underline{R}) \in \text{mod-}RG$  is called the homology representation of  $G$  on  $\mathcal{P}$ . However, it seems natural to go with the signs, from what we shall see shortly.

We have pointed out that, for each  $\mathfrak{M} \in \text{mod-}R\mathcal{P} \rtimes G$ ,  $LK_{\pi}(\mathfrak{M}) \cong \varinjlim_{\mathcal{P}} \mathfrak{M} \cong H_0(\mathcal{P}; \mathfrak{M}) \in \text{mod-}RG$ . Going up to the bounded derived categories, the functor  $\mathbb{L}K_{\pi}$  sends  $\kappa_M \in D^b(\text{mod-}R\mathcal{P} \rtimes G)$  to (a finite complex)  $\mathbb{C}_*(\mathcal{P}; \kappa_M) = \mathbb{C}_*(\mathcal{P}; \underline{R}) \otimes M \in D^b(\text{mod-}RG)$  [9]. Then

$$lk_{\pi}([\kappa_M]) = [\mathbb{C}_*(\mathcal{P}; \kappa_M)] = H_{\mathcal{P}}(G; \kappa_M) = H_{\mathcal{P}}(G)[M] \in G_0(RG).$$

It matches our definition of  $lk_{\pi}$  in last section.

Let  $G$  be a finite group,  $p$  be a prime that divides the order of  $G$ , and  $R$  be a field. Consider the  $G$ -posets of all  $p$ -subgroups  $\mathcal{S}_p^1$  and of all non-identity  $p$ -subgroups  $\mathcal{S}_p$ . The inclusion  $\mathcal{S}_p \hookrightarrow \mathcal{S}_p^1$  induces a long exact sequence of  $RG$ -modules involving relative homology

$$\cdots \rightarrow H_i(\mathcal{S}_p; \underline{R}) \rightarrow H_i(\mathcal{S}_p^1; \underline{R}) \rightarrow H_i(\mathcal{S}_p^1, \mathcal{S}_p; \underline{R}) \rightarrow \cdots.$$

Since  $\mathcal{S}_p^1$  has an initial object and is contractible, we have an equation

$$-\sum_i (-1)^i [H_i(\mathcal{S}_p^1, \mathcal{S}_p; \underline{R})] = \text{St}_{\mathcal{S}_p}(G) \in G_0(RG).$$

There is a so-called *generalized Steinberg module of  $G$  at  $p$*

$$\text{St}_p(G) = \sum_i (-1)^i [\mathbb{C}_i(\mathcal{S}_p; \underline{R})] - [R] \in a(RG),$$

When  $R$  has characteristic  $p$ ,  $\text{St}_p(G)$  is proved by Webb to be afforded by an element of  $D^b(\text{proj-}RG)$ , see [1, 5]. In fact, Webb showed that the augmented complex

$$\tilde{\mathbb{C}}_*(\mathcal{S}_p; \underline{R}) = \mathbb{P}_* \in D^b(\text{mod-}RG),$$

where  $\mathbb{P}_*$  is a complex of projective modules. Since

$$[\tilde{\mathbb{C}}_*(\mathcal{S}_p; \underline{R})] = \sum_i (-1)^i [H_i(\mathcal{S}_p; \underline{R})] - [R] \in G_0(RG),$$

it is reasonable to abuse the notation and identify  $\text{St}_p(G) = \text{St}_{\mathcal{S}_p}(G)$ .

For an arbitrary finite group  $G$ , suppose  $\mathcal{P} = \mathcal{B}_p$  is the  $G$ -poset of  $p$ -subgroups  $P$  satisfying  $P = O_p(N_G(P))$ . One may replace  $(\mathcal{S}_p^1, \mathcal{S}_p)$  by the pair  $(\mathcal{B}_p^1, \mathcal{B}_p)$ , because the subposets  $\mathcal{B}_p$  is  $G$ -homotopy equivalent to  $\mathcal{S}_p$ . There are other subposets that may be used for this purpose, see [1, ?, 5]. It means for instance  $\text{St}_{\mathcal{S}_p}(G) = \text{St}_{\mathcal{B}_p}(G)$ .

After recalling and rewriting facts, we summarize in the next result which clearly states why homology representations come from an induction process and naturally occur in representation theory.

**Lemma 4.2.** *Let  $G$  be a finite group,  $p$  be a prime that divides  $|G|$ ,  $\mathcal{P}$  be a finite  $G$ -poset,  $R$  be a field, and  $M$  be a right  $RG$ -module. Then*

$$lk_\pi([\kappa_M]) = [\mathbb{C}_*(\mathcal{P}; \underline{R})][M] = H_{\mathcal{P}}(G)[M] \in G_0(RG).$$

*It follows that*

$$lk_\pi r_\pi([M]) = lk_\pi([\kappa_M]) = [M] + \text{St}_{\mathcal{P}}(G)[M].$$

*or*

$$(lk_\pi r_\pi - \text{Id})([M]) = \text{St}_{\mathcal{P}}(G)[M].$$

*Particularly, when  $\mathcal{P} = \mathcal{S}_p$ , the operator  $lk_\pi r_\pi - \text{Id} : G_0(RG) \rightarrow G_0(RG)$  maps  $[R]$  to  $\text{St}_p(G)$ , the generalized Steinberg module at  $p$ .*

It rings a bell when one compares the last statement with a property of the Alvis-Curtis duality:  $D_G([\underline{R}]) = \text{St}_G$  (when  $R$  has characteristic zero, see [3, 4] and the following example), in the representation theory of finite groups of Lie type, where  $\text{St}_G$  is the usual Steinberg module (see Example 4.3 below).

**Example 4.3.** Let  $G$  be a finite Chevalley group of rank  $|S| > 1$  in characteristic  $p$  (here  $(W, S)$  stands for the Weyl group and its set of distinguished generators), see [3] for background. In this case, we consider  $\mathcal{B}_p$ , the subposet that is  $G$ -homotopy equivalent to  $\mathcal{S}_p$ . Borel and Tits showed that  $B\mathcal{B}_p$  is  $G$ -homotopy equivalent to the Tits building  $\Delta$ . In fact it is known that  $\mathcal{B}_p \cong \text{sd } \Delta$  is the barycentric subdivision of  $\Delta$  [1]. Assume  $R$  is a commutative ring with identity. Hence

$$[\mathbb{C}_*(\mathcal{B}_p; \underline{R})] = [\mathbb{C}_*(\text{sd } \Delta, R)] = [\mathbb{C}_*(\Delta, R)].$$

The  $i$ -simplices of  $\Delta$  are exactly all the parabolic subgroup  ${}^g P_I$ ,  $g \in G$ , with  $|I| = |S| - i - 1$ , and the stabilizers of these simplices are exactly the (proper) standard parabolic subgroups  $P_I$ ,  $I \subsetneq S$ . Involving only permutation modules, it follows that the above equals

$$\sum_i (-1)^i [\mathbb{C}_i(\Delta, R)] = \sum_{\substack{i=|S|-|I|-1 \\ I \subsetneq S}} (-1)^i [R \uparrow_{P_I}^G] = \sum_{\substack{i=|S|-|I|-1 \\ I \subsetneq S}} (-1)^i [R \uparrow_{P_I}^G],$$

in  $G_0(RG)$ , which gives rise to an equation

$$[\mathbb{C}_*(\mathcal{B}_p; \underline{R})] - [R] = (-1)^{|S|-1} \sum_{I \subsetneq S} (-1)^{|I|} [R \uparrow_{P_I}^G].$$

Now suppose  $(K, \mathcal{O}, k)$  is a  $p$ -modular system. By Lemma 4.2, for  $R = k$ ,

$$\text{St}_p(G) = [\mathbb{C}_*(\mathcal{B}_p; \underline{k})] - [k] = (-1)^{|S|-1} \sum_{I \subsetneq S} (-1)^{|I|} [k \uparrow_{P_I}^G].$$



In the representation theory of finite groups of Lie type, the usual *Steinberg module*  $\text{St}_G := \sum_{I \subseteq S} (-1)^{|I|} [K \uparrow_{P_I}^G]$ , for a Chevallay group, is known to be afforded by  $H_{|S|-1}(\Delta, K)$  [3]. By Lemma 4.2 again, for  $R = K$ ,

$$(lk_\pi r_\pi - \text{Id})([K]) = (-1)^{|S|-1} \text{St}_G.$$

In fact, upon reduction modulo  $p$ ,  $\text{St}_G$  becomes a projective simple  $kG$ -module  $\text{St}_p = H_{|S|-1}(\Delta, k)$ . Under the circumstance,  $\text{St}_p(G) = (-1)^{|S|-1} \text{St}_p$ .

We shall continue to pursue this point in the next section.

Now let us return to Lemma 4.2. When  $lk_\pi$  (or  $lk'_\pi$  if applicable) is surjective, then we may recover the ring  $G_0(RG)$  (or  $G_0^{st}(RG)$ ) as the image of  $G_0(R\mathcal{P} \rtimes G)$  (or  $G_0^{st}(R\mathcal{P} \rtimes G)$ ).

For example, if  $\mathcal{P}$  is connected and has vanishing homology (e.g. contractible), then for each  $RG$ -module  $M$ ,

$$lk_\pi r_\pi([M]) = [M] \in G_0(RG), \quad \text{for all } M \in \text{mod-}RG.$$

It implies that  $lk_\pi : G_0(R\mathcal{P} \rtimes G) \rightarrow G_0(RG)$  is surjective and that  $r_\pi : G_0(RG) \rightarrow G_0(R\mathcal{P} \rtimes G)$  is injective. Same claims can be made for  $lk'_\pi$  and  $r'_\pi$  between the stable Grothendieck rings. It happens, for instance, when  $\mathcal{P}$  is  $\mathcal{S}_p^1$ , or  $\mathcal{S}_p$  if  $O_p(G) \neq 1$ .

Another example is as follows. If  $R$  is a field of characteristic  $p$  and  $\mathcal{P} = \mathcal{S}_p$  or  $\mathcal{B}_p$ , then  $\text{St}_{\mathcal{P}}(G) = \text{St}_p(G) = 0 \in G_0^{st}(RG)$ . Thus the equation in Lemma 4.2 reads as

$$lk'_\pi r'_\pi([M]) = [M] \in G_0^{st}(RG), \quad \text{for all } M \in \text{mod-}RG.$$

Then  $lk'_\pi : G_0^{st}(R\mathcal{P} \rtimes G) \rightarrow G_0^{st}(RG)$  is surjective and  $r'_\pi : G_0^{st}(RG) \rightarrow G_0^{st}(R\mathcal{P} \rtimes G)$  is injective.

Based on the above observations, we record the following results.

**Theorem 4.4.** *Let  $\mathcal{P}$  be a finite  $G$ -poset.*

- (1) *If  $\mathcal{P}$  is connected and has vanishing homology, then both  $r_\pi$  and  $r'_\pi$  are injective. Meanwhile both  $lk_\pi$  and  $lk'_\pi$  are surjective.*
- (2) *If  $\text{St}_{\mathcal{P}}(G)$  is virtual projective, then  $r'_\pi$  is injective. Meanwhile  $lk'_\pi$  is surjective.*

Under the above assumptions, our result means that, to a large extent, the representation theory of groups is embedded into that of transporter categories. In another paper [11], we will show that many constructions and results in the local representation theory of finite groups can be extended to finite transporter categories.

In light of the above theorem, it will be interesting to compute  $lk_\pi([S])$  for each simple  $R\mathcal{P} \rtimes G$ -module. Regarded as a functor,  $S$

must be atomic, in the sense there exists an object  $x \in \text{Ob } \mathcal{P} \rtimes G$  such that  $S_{x,V}(y) = 0$  for all  $x \not\cong y$  in  $\text{Ob } \mathcal{P} \rtimes G$ . Moreover, as a functor we must have  $S(z) \cong S(x)$ , if  $z \cong x$ , as  $RG_x$ -modules. Here  $G_x = \text{Aut}_{\mathcal{P} \rtimes G}(x)$  is the stabilizer of  $x$  in  $G$ . Using techniques developed by Bouc, Oliver, Quillen, Thevenaz, Webb et al [1], we deduce that

$$\begin{aligned}
lk_\pi([S]) &= \sum_i (-1)^i [H_i(\mathcal{P}; S)] \\
&= \sum_i (-1)^i [\sum_{y \cong x} H_i(\mathcal{P}_{\geq y}, \mathcal{P}_{> y}; S(y))] \\
&= \sum_i (-1)^i [H_i(\mathcal{P}_{\geq x}, \mathcal{P}_{> x}; S(x))] \uparrow_{G_x}^G \\
&= \sum_i (-1)^i ([H_i(\mathcal{P}_{\geq x}, \mathcal{P}_{> x}; k)][S(x)]) \uparrow_{G_x}^G \\
&= \sum_i (-1)^i ([\tilde{H}_{i-1}(\mathcal{P}_{> x}; k)][S(x)]) \uparrow_{G_x}^G.
\end{aligned}$$

The last equality is true because  $\mathcal{P}_{\geq x}$  has an initial object and hence vanishing homology. Note that if  $x$  is maximal, then  $\mathcal{P}_{> x} = \emptyset$ , which has  $k$  as its -1 degree reduced homology and zero elsewhere. It means when  $x$  is maximal,  $lk_\pi([S]) = [S(x)] \uparrow_{G_x}^G$ .

Since for each simple module  $S$ , we always have  $lk_\pi([S]) \in G_0(RG_x) \uparrow_{G_x}^G$  for some  $x$ , when  $lk_\pi$  (or  $lk'_\pi$  if applicable) is surjective, we get the following “induction theorems”.

**Corollary 4.5.** *Let  $\mathcal{P}$  be a finite  $G$ -poset.*

(1) *If  $\mathcal{P}$  is connected and has vanishing homology, then*

$$G_0(RG) = \sum_{[x] \subset \text{Ob}(\mathcal{P} \rtimes G)} G_0(RG_x) \uparrow_{G_x}^G.$$

and

$$G_0^{st}(RG) = \sum_{[x] \subset \text{Ob}(\mathcal{P} \rtimes G)} G_0^{st}(RG_x) \uparrow_{G_x}^G.$$

(2) *If  $\text{St}_{\mathcal{P}}(G)$  is virtual projective, then*

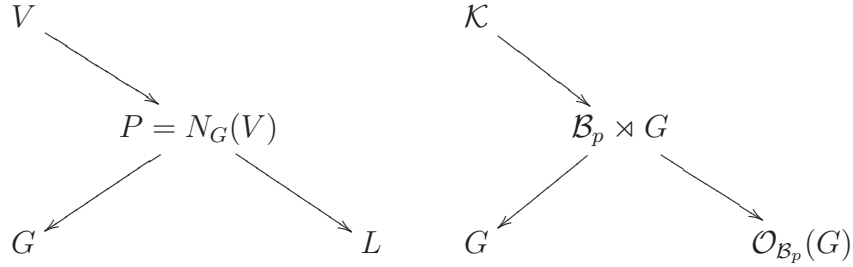
$$G_0^{st}(RG) = \sum_{[x] \subset \text{Ob}(\mathcal{P} \rtimes G)} G_0^{st}(RG_x) \uparrow_{G_x}^G.$$

## 5. ORBIT CATEGORIES AND THE ALVIS-CURTIS DUALITY

Motivated by Lemma 4.2 and Example 4.3, we try to give a new construction of the Alvis-Curtis duality, using local categories. We are particularly interested in the orbit categories. Throughout this section, we suppose  $G$  is a Chevalley group. Consider  $\mathcal{B}_p$ , the transporter

category  $\mathcal{B}_p \rtimes G$  and the resulting orbit category  $\mathcal{O}_{\mathcal{B}_p}(G)$ . The automorphism groups in the preceding orbit category are exactly the Levi subgroups of  $G$ . We will see that, in this situation, both the left and right Kan extensions need to be involved.

In the classical case, there are the Harish-Chandra restriction (also called the truncation) and the Harish-Chandra induction (also called the generalized induction) relating  $kG$ -mod with  $kL$ -mod, for  $L$  a Levi subgroup. It may be pictured as the left diagram.



Replacing the group extension by a category extension, the diagram on the right becomes a categorical version of the left whose automorphism groups are exactly the Levi subgroups of  $G$ . The functor  $\mathcal{K}$  maps each object  $V \in \text{Ob}(\mathcal{B}_p \rtimes G)$  to  $V$  itself, and can be identified with the set of unipotent subgroups.

Recall that we regard  $\mathcal{P} \rtimes G$  as generalized subgroups of  $G$ . We wish to lift the previously mentioned Harish-Chandra restriction and induction to functors relating representations of  $G$  and  $\mathcal{O}_{\mathcal{B}_p}(G)$ , as an “amalgam” of Levi subgroups. It is certainly motivated by our work in the preceding section which utilized the relation between  $G$  and  $\mathcal{B}_p \rtimes G$  to obtain the generalized Steinberg module.

In this part, we shall be working over  $R = \mathbb{C}$ . Given a finite EI category  $\mathcal{C}$  [6], every right  $\mathbb{C}\mathcal{C}$ -module is of finite projective dimension. Thus

$$G_0(\mathbb{C}\mathcal{C}) = G_0(D^b(\text{mod-}\mathbb{C}\mathcal{C})) = G_0(D^b(\text{proj-}\mathbb{C}\mathcal{C})) \cong K_0(\mathbb{C}\mathcal{C}).$$

For consistency, we shall stick with the notation  $G_0(\mathbb{C}\mathcal{C})$ .

From the canonical functors  $G \leftarrow \mathcal{B}_p \rtimes G \rightarrow \mathcal{O}_{\mathcal{B}_p}(G)$ , we obtain functors between module categories

$$\text{Tr}_G = RK_p \text{Res}_\pi : \text{mod-}\mathbb{C}G \xrightarrow{\text{Res}_\pi} \text{mod-}\mathbb{C}\mathcal{B}_p \rtimes G \xrightarrow{RK_p} \text{mod-}\mathbb{C}\mathcal{O}_{\mathcal{B}_p}(G),$$

and

$$\text{I}_G = LK_\pi \text{Res}_p : \text{mod-}\mathbb{C}G \xleftarrow{LK_\pi} \text{mod-}\mathbb{C}\mathcal{B}_p \rtimes G \xleftarrow{\text{Res}_p} \text{mod-}\mathbb{C}\mathcal{O}_{\mathcal{B}_p}(G).$$

The lower functor  $I_G = LK_\pi \text{Res}_p$ , applied to any *atomic* module (a.k.a. an atomic functor on  $\mathcal{O}_{\mathcal{B}_p}(G)$ ), and equivalently a module of a Levi subgroup), is the same as the Harish-Chandra induction. By the adjunctions between the restriction and the Kan extensions, we readily verify that

$$\text{Hom}_{\mathbb{C}G}(I_G(\mathfrak{M}), N) \cong \text{Hom}_{\mathbb{C}\mathcal{O}_{\mathcal{B}_p}}(\mathfrak{M}, T_G(N))$$

which means that  $I_G$ , called the *generalized induction*, is the left adjoint of  $T_G$ , called the *generalized truncation*.

Let us exploit the basic properties of these two new functors that we have just introduced. To understand  $T_G$ , we must calculate  $RK_p$ . In fact, as we have seen in [8], since  $\mathcal{B}_p \rtimes G \rightarrow \mathcal{O}_{\mathcal{B}_p}(G)$  is part of a category extension, for every  $\mathfrak{N} \in \text{mod-}\mathbb{C}\mathcal{B}_p \rtimes G$ ,

$$RK_p(\mathfrak{N}) = \varprojlim_{\mathcal{K}} \mathfrak{N} \cong H^0(\mathcal{K}; \mathfrak{N}).$$

At an object  $V \in \text{Ob } \mathcal{B}_p$ ,  $RK_p(\mathfrak{N})(V) = H^0(V, \mathfrak{N}(V)) = \mathfrak{N}(V)^V$ . If there is a morphism  $V \rightarrow V'$ , one easily defines  $\mathfrak{N}(V')^{V'} \rightarrow \mathfrak{N}(V)^V$ . Under the circumstance, the functor  $RK_p$  is exact, and hence so is  $T_G$ . It follows that  $I_G$  is right exact. The induced map  $rk_p : G_0(\mathbb{C}\mathcal{B}_p \rtimes G) \rightarrow G_0(\mathbb{C}\mathcal{O}_{\mathcal{B}_p}(G))$  is naturally given by

$$[\mathfrak{N}] \mapsto [RK_p \mathfrak{N}].$$

Note that, when  $\mathfrak{N} = \kappa_M = \text{Res}_\pi(M) \in \text{mod-}\mathbb{C}\mathcal{B}_p \rtimes G$  for some  $\mathbb{C}G$ -module  $M$ ,  $H^0(\mathcal{K}; \kappa_M) = T(M)$  may be regarded as a fixed-point coefficient system  $\mathcal{F}_M$  on the Tits building, in the sense of Ronan-Smith [1, 5].

The two functors  $I_G$  and  $T_G$  induce derived functors  $\mathbb{I}_G$  and  $\mathbb{T}_G$ , and the following maps between the Grothendieck groups

$$i_G : G_0(\mathcal{O}_{\mathcal{B}_p}(G)) \rightarrow G_0(\mathbb{C}G); \quad [\mathfrak{M}] \mapsto \sum_i (-1)^i [H_i(\mathcal{B}_p; \mathfrak{M})],$$

and

$$t_G : G_0(\mathbb{C}G) \rightarrow G_0(\mathbb{C}\mathcal{O}_{\mathcal{B}_p}(G)); \quad [M] \mapsto [H^0(\mathcal{K}; \kappa_M)].$$

**Theorem 5.1.** *Suppose  $G$  is a Chevalley group of type  $(W, S)$  such that  $|S| > 1$ . Then*

$$D_G := i_G t_G - \text{Id} : G_0(\mathbb{C}G) \rightarrow G_0(\mathbb{C}G)$$

*is the Alvis-Curtis duality, up to a sign.*

*Proof.* We need to compute

$$i_G t_G([M]) = i_G([H^0(\mathcal{K}; \kappa_M)]) = \sum_i (-1)^i [H_i(\mathcal{B}_p; H^0(\mathcal{K}; \kappa_M))].$$

Let  $\Delta$  be the Tits building. The  $i$ -simplices are exactly all the parabolic subgroup  ${}^gP_I$  with  $|I| = |S| - i - 1$ , and  $\mathcal{B}_p \cong \text{sd } \Delta$ . If we consider the fixed-point coefficient system  $\mathcal{F}_M$ , of Ronan-Smith, then its restriction along  $\text{sd } \Delta \rightarrow \Delta$  gives rise to  $H^0(\mathcal{K}; \kappa_M)$ . Thus

$$\sum_i (-1)^i [H_i(\mathcal{B}_p; H^0(\mathcal{K}; \kappa_M))] = \sum_i (-1)^i [C_i(\mathcal{B}_p; H^0(\mathcal{K}; \kappa_M))]$$

equals

$$\sum_i (-1)^i [C_i(\text{sd } \Delta; H^0(\mathcal{K}; \kappa_M))] = \sum_i (-1)^i [C_i(\Delta; \mathcal{F}_M)].$$

But

$$C_i(\Delta; \mathcal{F}_M) \cong \bigoplus_{\substack{|I|=|S|-i-1 \\ I \subsetneq S}} M^{V_I} \uparrow_{P_I}^G.$$

and then

$$\begin{aligned} [C_*(\Delta; \mathcal{F}_M)] - [M] &= \sum_{i=|S|-|I|-1} (-1)^i [M^{V_I} \uparrow_{P_I}^G] \\ &= (-1)^{|S|-1} \sum_{I \subsetneq S} (-1)^{|I|} [M^{V_I} \uparrow_{P_I}^G]. \end{aligned}$$

It implies that

$$D_G([M]) = i_G t_G([M]) - [M] = (-1)^{|S|-1} \sum_{I \subsetneq S} (-1)^{|I|} [M^{V_I} \uparrow_{P_I}^G].$$

is exactly the Alvis-Curtis duality, up to the sign  $(-1)^{|S|-1}$ .  $\square$

Combine with Lemma 4.2 and Example 4.3, we do have  $D_G(\mathbb{C}) = \text{St}_G$ . However, we are unable to provide an intrinsic proof of  $D_G^2 = \text{Id}$ , although we suspect that it comes directly from the adjunction between functors, say  $I_G$  and  $T_G$ .

## REFERENCES

- [1] Benson, D. J., *Representations and Cohomology II*, Cambridge Studies in advanced mathematics **31**, Cambridge (1998).
- [2] Cabanes, E., Rickard, J., *Alvis-Curtis duality as an equivalence of derived categories*, in: *Modular Representation Theory of Finite Groups*, Ed. M. Collins, B. Parshall, L. Scott, de Gruyter (2001), pp 157-174.
- [3] Curtis, C., Reiner, I., *Methods of Representation Theory II*, John Wiley & Sons (1987).
- [4] Digne, F., Michel, J., *Representations of Finite Groups of Lie Type*, LMS Student Texts **21**, Cambridge University Press (1991).
- [5] Smith, S. D., *Subgroup Complexes*, AMS (2013).
- [6] Webb, P. J., *An introduction to the representations and cohomology of categories*, in: *Group Representation Theory*, EPFL Press (2007), pp 149-173.

- [7] Witherspoon, S., *The ring of equivariant vector bundles on finite sets*, J. Algebra **175** (1994), 274–286.
- [8] Xu, F., *On local categories of finite groups*, Math. Z. **272** (2012), 1023–1036.
- [9] Xu, F., *Becker-Gottlieb transfer for Hochschild cohomology*, Proc. Amer. Math. Soc. **142** (2014), 2593–2608.
- [10] Xu, F., *Spectra of tensor triangulated categories over category algebras*, Arch. Math. **103** (2014), 235–253.
- [11] Xu, F., *Local representation theory of finite transporter categories*, preprint.
- [12] Zimmermann, A., *Representation Theory*, Algebra and Application **19**, Springer (2013).

*E-mail address:* fxu@stu.edu.cn

DEPARTMENT OF MATHEMATICS, SHANTOU UNIVERSITY, SHANTOU, GUANGDONG 515063, CHINA